English the fourth German edition of his *Elements of the Theory of Functions of a Complex Variable* (Philadelphia, 1896) * Anton Winckler (died, 1892), professor of higher mathematics in the polytechnic school of Vienna, was born August 3 * Wilhelm Johann Otto Ligowski (died, 1893), teacher of mathematics in the marine academy and marine school of Kiel, was born August 10. He is the author of *Tafeln der Hyperbelfunctionen und der Kreisfunctionen* (Berlin, 1890), and of *Sammlung fünfstelliger logarithmischer, trigonometrischer und nautischer Tafeln nebst Erklärungen und Formeln der Astronomie* (Kiel, 1873; fourth edition, Berlin, 1900) * The celebrated English mathematician Arthur Cayley (died, 1895) was born August 16. He was a prolific author of mathematical memoirs * The noted German physicist, Hermann Ludwig Ferdinand Helmholtz (died, 1894), was born August 31 * First number of the *Astronomische Nachrichten* published in September * Franz Friedrich Ernst Brünnow (died, 1891), German astronomer, was born November 18. He was director of the observatory at Ann Arbor, Mich., 1854–1864, professor of astronomy at the University of Dublin and astronomer royal for Ireland, 1865–1874 * Astronomical observatory at Cape of Good Hope founded * Samuel Vince, Plumian professor of astronomy and experimental philosophy at Cambridge University, 1796–1821, died. He was the author of several books including *The Elements of Conic Sections* (1781), and *A Treatise on Plane and Spherical Trigonometry* (1800) * *Cours d'analyse de l'Ecole Polytechnique,* prèmiere partie, by A. L. Cauchy (1789–1857), was published; no other part appeared * Published at Rome, Abate Pietro Franchini's *Saggio sulla Storia delle Matematiche, corredato di scelte Notizie Biografiche,* volume 1 * Published at Mainz, second enlarged edition of J. J. I. Hoffmann's *Der Pythagorische Lehrsatz, mit 32 theils bekannten, theils neuen Beweisen versehen* * Published at London, *Elementary Illustrations of the Celestial Mechanics of Laplace,* Part I, by Thomas Young (1773–1829), one of the "most eminent physicists of his time" * Published at Edinburgh, Sir John Leslie's *Geometrical Analysis, and the Geometry of Curve Lines, being volume second of a course of Mathematics, and designed as an Introduction to the Study of Natural Philosophy.* This work was a development of part of *Elements of Geometry, Geometrical Analysis and Plane Trigonometry, with an Appendix, Notes, and Illustrations,* Edinburgh, 1809 (second edition, 1811; French translation, edited by Hachette, 1818) * Published at Haarlem, James Lockhart's *Leerwijze om den Cubik-Wortel uit alle Getallen te trekken.*

SOME ARITHMETIC OPERATIONS WITH TRANSFINITE ORDINALS.

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If we write out the first few non-negative integers in order of increasing magnitude, we shall have

$$0,\ 1,\ 2,\ \cdots,\ (n-1),\ n.$$

This finite ordered sequence constitutes an example of an order type to which the name, $n + 1$, may be given. This is also the name of the first ordinal integer following all of those of the given set. This statement applied inductively accounts formally for all positive integers. It is also available for the introduction of transfinite ordinals. For example the sequence, $0, 1, 2, \cdots, n, \cdots$, including zero and all positive finite integers is an example of an order type to which the name, ω ("Omega"), may be given. Next after ω follows $\omega + 1$, and so forth. Two successive sets of order type ω when considered together constitute an example of the order type called $\omega 2$, but not 2ω, for reasons that we shall see. The order types $\omega 3$, $\omega 4$, etc., are obtained analogously, finally giving $\omega\omega$, called also ω^2, and so forth.

Georg Cantor, to whom is due the credit for introducing these transfinite ordinals on a rigorous basis, defined also the addition and multiplication of ordinals. The ordered sum, $a + b$ of two ordinals, a and b, is the ordinal whose order type is obtainable by placing after a sequence of the order type of a, one of the order type of b. Thus $(\omega) + (1) = \omega + 1$, $1 + \omega = \omega$, $\omega + \omega = \omega 2$, $(m) + (n) = m + n$ where m and n are finite ordinals. The ordered product ab of two ordinals, a and b, is the ordinal whose order type is obtainable by replacing each element of b by a set of the order type of a.[1] Thus 3×4 may be represented as $(\cdots)\ (\cdots)\ (\cdots)\ (\cdots)$, while 4×3 would be represented by $(\cdots\cdot)\ (\cdots\cdot)\ (\cdots\cdot)$. We see also that $(\omega)(2) = \omega 2$, and more generally $(\omega)(n) = \omega n$, while $n\omega = \omega$. The following may be verified,

$$(\omega + n)(\omega + m) = \omega^2 + \omega m + n, \qquad a(b + c) = ab + ac,$$

where m, n, $(m > 0)$ are finite ordinals, and a, b, c, any ordinals. Propositions *not* in general valid, are: $a + b = b + a$, $ab = ba$, and $(a + b)c = ac + bc$.

While addition and multiplication are always possible among ordinals, subtraction is certainly not in general admissible. Thus there is no ordinal x to satisfy the relation $x + 1 = \omega$ so that $\omega - 1$ cannot be itself an ordinal, if it is to represent an x satisfying the equation referred to. On the other hand there is an x, namely ω, for which $1 + x = \omega$. The noncommutative character of addition is what causes part of the difficulty. The following general fact may be noted. If a and b be two ordinals, and $b > a$, then there is one and only one ordinal x satisfying the equation $a + x = b$. This may be expressed in the statement that if $a < b$, $-a + b$ is a uniquely defined ordinal.

An interesting consequence of the uniqueness of left hand subtraction is the following theorem, which is believed to be new:

The highest common left hand divisor of two ordinals may be found by Euclid's algorithm.

The process may be illustrated as follows in the case of

$$\omega^3 + \omega^2 + \omega 5 + 3, \qquad \text{and} \qquad \omega^3 + \omega^2 7 + \omega 2 + 3.$$

[1] For this notation, as well as for further references, compare the bibliography in E. V. Huntington, *The Continuum and other Types of Serial Order*, second edition, 1917, p. 74.

Step	Dividend	Divisor	Remainder	Quotient
1	$\omega^3 + \omega^2 7 + \omega 2 + 3$	$\omega^3 + \omega^2 + \omega 5 + 3$	$\omega^2 6 + \omega 2 + 3$	1
2	$\omega^3 + \omega^2 + \omega 5 + 3$	$\omega^2 6 + \omega 2 + 3$	$\omega^2 + \omega 5 + 3$	ω
3	$\omega^2 6 + \omega 2 + 3$	$\omega^2 + \omega 5 + 3$	$\omega^2 + \omega 2 + 3$	5
4	$\omega^2 + \omega 5 + 3$	$\omega^2 + \omega 2 + 3$	$\omega 3 + 3$	1
5	$\omega^2 + \omega 2 + 3$	$\omega 3 + 3$	$\omega 2 + 3$	ω
6	$\omega 3 + 3$	$\omega 2 + 3$	$\omega + 3$	1
7	$\omega 2 + 3$	$\omega + 3$	0	2

Thus $\omega + 3$ is the greatest common left hand divisor. Indeed the original expressions may be written respectively as $(\omega+3)(\omega^2+\omega+5)$ and $(\omega+3)(\omega^2+\omega 7+2)$, where the greatest common left hand divisor of $\omega^2 + \omega + 5$ and $\omega^2 + \omega 7 + 2$ is 1. While ω contains each finite ordinal as a left-hand divisor, $\omega + 1$ is prime to them all.

The question remains at this stage as to whether the system of ordinals may be so extended that for each quantity of the system, the negative of the quantity also is in the system, and where addition and multiplication shall still be possible in all cases. If a somewhat generous interpretation be given to these terms, as necessitated by the fact that addition is not ordinarily commutative, the answer is, yes. This will now be demonstrated by the introduction of what appears to be a new concept in this connection.

We shall consider a system whose elements may be formally written as $a - b$, where a and b are any ordinals. The expression $a - b$ will be regarded as the formal sum of a sequence of order type a, every element of which is $+ 1$ followed by a sequence of a reverse order type, namely the reverse of b, in which, furthermore, each element is $- 1$. Thus if $a = \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$ we shall write $- a$ as $- n_0 - \omega n_1 - \omega^2 n_2 - \omega^3 n_3$. The fundamental principle to be observed is that addition (among a finite number of terms of the same sign) is throughout associative. Any combination $- a + a$ may be replaced by zero, while $a - a$ will not in general be regarded as the same as zero. If $a = \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$, then $- a + a$ may be written as $- n_0 - \omega n_1 - \omega^2 n_2 - \omega^3 n_3 + \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$. Pairs of terms may be dropped out successively from the center until zero results.

To form the sum $(a_1 - b_1) + (a_2 - b_2)$, compare the ordinals a_2 and b_1. If a_2 be equal to or greater than b_1 it may be written as $a_2 = b_1 + c$, giving $a_1 - b_1 + b_1 + c - b_2$ which reduces to $(a_1 + c) - b_2$. If on the other hand b_1 is equal to or greater than a_2 we may write $b_1 = a_2 + d$, or $- b_1 = - d - a_2$ whence the sum becomes $a_1 - d - a_2 + a_2 - b_2$ or $a_1 - (b_2 + d)$.

Before considering multiplication of these expressions, we shall return to note a feature of ordinals. Positive ordinals are of two types, *terminating* and *non-terminating*. While $\omega + 1$, $\omega 2 + 7$, 8, each is an order type with a last element, ω, $\omega^2 + \omega 2$ and ω^ω each is an order type with no last element. The sum $a + b$ of two elements is found to be terminating if and only if the second quantity, b, is terminating, the product ab is terminating if and only if both a and b are ter-

minating. An infinite terminating ordinal, a, can be expressed uniquely as a sum $a' + n$ where n is finite, and a' is nonterminating. The finite term n is called the *termination* of a. In any product ab, where a is terminating with termination, n, the product will be terminating and have this same termination n, for every terminating b, although this termination, which is independent of the character of b so long as b is terminating, vanishes whenever b is nonterminating.

In the expression $a - b$, let l denote the lesser of the two quantities a, b, and d denote their difference. Then $a - b$ may be written $l \pm d - l$, where $\pm d$ is called the *stem*, and l, the *affix* of the expression, $a - b$. The sign of $\pm d$ depends upon the particular expression $a - b$ considered and is completely determined except for the trivial case where $d = 0$. The product $(l \pm d - l)c$ is defined as $l \pm dc - l$ when c is terminating, and as $\pm dc$ when c is nonterminating. For example

$$(\omega - 3)(\omega^2 + \omega 2 + 7) = 3 + \omega(\omega^2 + \omega 2 + 7) - 3 = \omega^3 + \omega^2 2 + \omega 7 - 3$$

$$(\omega - 3)(\omega^2 + \omega 2) = \omega(\omega^2 + \omega 2) = \omega^3 + \omega^2 2$$

$$\begin{aligned}((\omega + 1) - 4)(\omega^2 + \omega 2 + 7) &= 4 + (\omega + 1)(\omega^2 + \omega 2 + 7) - 4 \\ &= \omega^3 + \omega^2 2 + \omega 7 + 1 - 4\end{aligned}$$

$$((\omega + 1) - 4)(\omega^2 + \omega 2) = (\omega + 1)(\omega^2 + \omega 2) = \omega^3 + \omega^2 2$$

The above rule has the desirable though not essential property that if we identify $a - b$ and $(a + n) - (b + n)$, then we may also identify $(a - b)c$ with $((a + n) - (b + n))c$, where n is a finite ordinal. Thus so far as finite ordinals are concerned we may if we desire regard $n - n$ as identical with zero, although this is not desirable in some applications.

The product $(a_1 - b_1)(a_2 - b_2)$ is defined as $(a_1 - b_1)a_2 + (b_1 - a_1)b_2$. The product $(a_1 - b_1)((a_2 - b_2) + (a_3 - b_3))$ is found to be equal to $(a_1 - b_1)(a_2 - b_2) + (a_1 - b_1)(a_3 - b_3)$ if the stem of $a_1 - b_1$ is positive. In case this stem is negative, we have instead $(a_1 - b_1)(a_3 - b_3) + (a_1 - b_1)(a_2 - b_2)$, as is to be expected.

It is to be noted that the above is closely suggestive of the process by which rational fractions are introduced in the development of the number system by means of pairs of integers.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

14. Maupertuis and Frederick the Great.

Among those whom Frederick the Great called to his court for the purpose of accomplishing what the Ptolemies had done for Alexandria, the caliphs for Bagdad, and the Medici for Florence, there were a few scientists and literati of genius, and still more of near ability. Of the latter, Maupertuis is perhaps the